WEAKLY COUPLED SYSTEMS OF FULLY NONLINEAR PARABOLIC EQUATIONS IN THE HEISENBERG GROUP

QING LIU AND XIAODAN ZHOU

Abstract. This paper is devoted to viscosity solutions to weakly coupled systems of fully nonlinear parabolic equations in the first Heisenberg group. We extend the known results in the Euclidean space to the Heisenberg group, including the uniqueness and existence of solutions with exponential growth at space infinity under monotonicity and other regularity assumptions on the parabolic operators. In addition, Lipschitz preserving properties of the system are also studied.

1. Introduction

In this paper, we discuss uniqueness, existence and Lipschitz regularity of viscosity solutions to a class of weakly coupled systems of fully nonlinear parabolic equations in the first Heisenberg group $\mathbb{H}$:

$$\begin{cases} \partial_t u_i + F_i (p, u_1, \ldots, u_m, \nabla_H u_i, (\nabla_H^2 u_i)^*) = 0 & \text{in } \mathbb{H} \times (0, \infty) \text{ for } i = 1, \ldots, m, \\ u_i(\cdot, 0) = h_i, & \text{in } \mathbb{H} \text{ for } i = 1, \ldots, m, \end{cases}$$

where $F_i : \mathbb{H} \times \mathbb{R}^m \times \mathbb{R}^2 \times S^2 \to \mathbb{R}$ and $h_i : \mathbb{H} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are given continuous functions satisfying the assumptions to be elaborated in a moment. Here $\nabla_H u$ and $(\nabla_H^2 u_i)^*$ respectively denote the horizontal gradient and the symmetrized horizontal Hessian of a function $u : \mathbb{H} \to \mathbb{R}$; consult Section 2 for precise definitions.

Weakly coupled systems of first or second order fully nonlinear equations in the Euclidean space were studied in a vast literature; see for example [7, 10, 17, 18, 16, 20] for well-posedness results and applications in optimal control and differential games. More recently, weakly coupled systems involving the $\infty$-Laplace with a stochastic game interpretation was addressed in [24]. Such nonlinear systems however are far less understood in more general spaces. Motivated by applications of viscosity solution theory in general geodesic spaces [13], in this work we generalize the known results on systems in the Heisenberg group, which is an important example of metric spaces whose geometry is much different from the Euclidean spaces or other Riemannian manifolds.

As for viscosity solution theory on sub-Riemannian manifolds, there has also been extensive study on various nonlinear equations such as the Hamilton-Jacobi equation [23], $p$-Laplace equation with $1 < p \leq \infty$ [3, 4, 11] and level set mean curvature flow equation [5, 12]; we refer the reader to [22] for a comprehensive introduction of the theory. Noticing that these results are restricted to single equations, we then aim to provide a complement in the case of monotone weakly coupled systems.

Date: August 11, 2017.
2010 Mathematics Subject Classification. 35R03, 35D40, 35K40.
Key words and phrases. Heisenberg group, viscosity solutions, weakly coupled parabolic systems.
Let us prepare more notations in order to present our main results. Let $S^k$ stand for the set of all $k \times k$ symmetric matrices. For simplicity, we denote by $u$ the vector-valued function $\mathbb{H} \times [0, \infty) \to \mathbb{R}^m$ with components $u_1, u_2, \ldots, u_m$. For any two vectors $a, b \in \mathbb{R}^m$, we write $a \leq b$ if the components satisfy $a_i \leq b_i$ for all $i = 1, 2, \ldots, m$. Hence, for any $(p, t) \in \mathbb{H} \times [0, \infty)$ and $u, v : \mathbb{H} \times [0, \infty) \to \mathbb{R}^m$, the notation $u(p, t) \leq v(p, t)$ means that $u_i(p, t) \leq v_i(p, t)$ holds for all $i = 1, 2, \ldots, m$.

We impose the following assumptions on the operators $F_i$, which are similar to those appearing in the Euclidean case (cf. [17]).

(A1) (Regularity) $F_i$ is Lipschitz continuous in $\mathbb{H} \times \mathbb{R}^m \times \mathbb{R}^2 \times S^2$ uniformly for all $i = 1, 2, \ldots, m$, i.e., there exists $L > 0$ such that

$$|F_i(p, r, \xi, X) - F_i(p', r', \xi', X')| \leq L(|p' - p| + |r - r'| + |\xi - \xi'| + |X - X'|),$$

for all $p, p' \in \mathbb{H}$, $r, r' \in \mathbb{R}^m$, $\xi, \xi' \in \mathbb{R}^2$, $X, X' \in S^2$ and $i = 1, 2, \ldots, m$.

(A2) (Monotonicity) For any $r, r' \in \mathbb{R}^m$, $p \in \mathbb{H}$, $\xi \in \mathbb{R}^2$ and $X \in S^2$, when

$$\max_j (r_j - r'_j) = r_i - r'_i \geq 0$$

for some $i = 1, 2, \ldots, m$, we have

$$F_i(p, r, \xi, X) \geq F_i(p, r', \xi, X).$$

(A3) (Parabolicity) For every $i = 1, 2, \ldots, m$ and for any $r, p \in \mathbb{H}$, $\xi \in \mathbb{R}^2$ and $X, Y \in S^2$, we have

$$F_i(p, r, \xi, X) \leq F_i(p, r, \xi, Y)$$

if $X \geq Y$.

As a concrete example, the semilinear parabolic system

$$\begin{cases}
\partial_t u_1 - \Delta_H u_1 + |\nabla_H u_1| + au_1 - bu_2 = 0 & \text{in } \mathbb{H} \times (0, \infty), \\
\partial_t u_2 - \Delta_H u_2 + |\nabla_H u_2| - cu_1 + du_2 = 0 & \text{in } \mathbb{H} \times (0, \infty), \\
u_i(\cdot, 0) = h_i, & \text{in } \mathbb{H} \text{ for } i = 1, 2
\end{cases}$$

satisfies all of the assumptions above, provided that $a \geq b \geq 0$ and $d \geq c \geq 0$. Here $\Delta_H u_i = \text{tr}(\nabla^2_H u_i)^*$ stands for the horizontal Laplacian of $u_i$ with $i = 1, 2$.

The assumption (A2) implies that

$$F_i(p, r_1, r_2, \ldots, r_m, \xi, X) \leq F_i(p, r_1 + c, r_2 + c, \ldots, r_m + c, \xi, X)$$

for any $c \geq 0$ and $i = 1, 2, \ldots, m$.

Under the assumptions (A1)–(A3), we investigate existence and uniqueness of solutions to (1.1)–(1.2) in the framework of viscosity solutions.

**Theorem 1.1** (Main result). Assume that (A1)–(A3) hold. Suppose that $h = (h_1, h_2, \ldots, h_m) : \mathbb{H} \to \mathbb{R}^m$ is continuous and satisfies the following growth condition:

(G0) there exists $k > 0$ and $C_0 > 0$ such that

$$|h(p)| \leq C_0 e^{k|p|}$$

for all $p \in \mathbb{H}$.

Then there exists a unique viscosity solution of (1.1)–(1.2).
The uniqueness of unbounded viscosity solutions follows from a comparison principle, as presented in Theorem 3.1, which is a generalization of the result on single fully nonlinear equations on $\mathbb{H}$. The main difficulty lies at the unboundedness of the domain. To this end, we restrict our comparison to solutions with exponential growth at space infinity and adapted the arguments in [9, 1] to our monotone system in the sub-Riemannian circumstances.

Although Perron’s method is expected to give the existence of solutions, as shown in [16] in the Euclidean space, our existence proof is partially based on an adaptation of deterministic discrete games introduced by Kohn and Serfaty [19]. It is well known that weakly coupled systems can be derived by including stochastic switching in the games [18, 24]. The game associated to our system (1.1)−(1.2) can actually be established in a similar way, but we choose not to show this in order to keep the deterministic flavor. Instead, we begin with the dynamic programming principle (as given precisely in (4.2)) and inductively use it to define an approximate solution $u^\varepsilon$ of (1.1)−(1.2).

The convergence of $u^\varepsilon$ to the unique solution $u$ of (1.1)−(1.2) streamlines the proof in [19]. We show that the relaxed limits of $u^\varepsilon$ are respectively a subsolution and a supersolution and adopt the comparison principle to conclude the proof. However, we here need extra work again to overcome the difficulty caused by unboundedness of the domain with the sub-Riemannian geometry. We construct upper and lower barrier functions to ensure that our scheme maintains the exponential growth in space; see Section 4 for details.

The last part of this paper is devoted to the Lipschitz preserving properties for (1.1)−(1.2). In contrast to the well-known fact that the Lipschitz continuity of the initial value is preserved for a large class of degenerate parabolic equations in the Euclidean space [15, 14], it is however not the case in $\mathbb{H}$ in general even for single equations; see Example 5.1. This is mainly due to the general lack of invariance for Lipschitz continuity under right translations in $\mathbb{H}$. Following [21], we therefore first show the preservation property of right-invariant Lipschitz continuity for (1.1)−(1.2) and discuss several special cases under additional assumptions on symmetry of solutions.

It is worth pointing out that the Lipschitz preserving result can be improved if the operators $F_i$ are reduced to the first order. In fact, without assuming any additional symmetry of solutions, we prove in Theorem 5.5 that the (left-invariant) Lipschitz continuity is preserved when the operator $F_i$ depends only on $u$ and the horizontal gradient $|\nabla_H u_i|$ for all $i = 1, 2, \ldots, m$.

The paper is organized in the following way. In Section 2, we give a brief review of the Heisenberg group $\mathbb{H}$ and the theory of viscosity solutions, especially the Crandall-Ishii lemma adapted to $\mathbb{H}$. We prove a comparison principle and show the uniqueness of unbounded solutions of (1.1)−(1.2) with exponential growth in Section 3. In Section 4 we construct a scheme, based on a game interpretation, to obtain the existence of solutions. The Lipschitz preserving properties are discussed in Section 5.

2. Preliminaries

2.1. A brief review of the Heisenberg group. Recall that the Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = (x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p)),$$
for all \( p = (x_p, y_p, z_p) \) and \( q = (x_q, y_q, z_q) \) in \( \mathbb{H} \). Note that the group inverse of \( p = (x_p, y_p, z_p) \) is \( p^{-1} = (-x_p, -y_p, -z_p) \). The Kórányi gauge is given by

\[
|p|_G = \left( (p_1^2 + p_2^2)^2 + 16p_3^2 \right)^{1/4},
\]

and the left-invariant Kórányi or gauge metric is

\[
d_L(p, q) = |p^{-1} \cdot q|_G.
\]

For our particular application later, we denote \( \mathbb{H}_0 = \{ p \in \mathbb{H} : p = (p_1, p_2, 0) \text{ for } p_1, p_2 \in \mathbb{R} \} \).

The Lie Algebra of \( \mathbb{H} \) is generated by the left-invariant vector fields

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}; \\
X_2 &= \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}; \\
X_3 &= \frac{\partial}{\partial z}.
\end{align*}
\]

One may easily verify the commuting relation \( X_3 = [X_1, X_2] = X_1X_2 - X_2X_1 \).

The horizontal gradient of \( u \) is given by

\[
\nabla_H u = (X_1u, X_2u)
\]

and the symmetrized second horizontal Hessian \( (\nabla^2_H u)^* \in S^2 \) is given by

\[
(\nabla^2_H u)^* := \begin{pmatrix}
X_1^2u & (X_1X_2u + X_2X_1u)/2 \\
(X_1X_2u + X_2X_1u)/2 & X_2^2u
\end{pmatrix}.
\]

A piecewise smooth curve \( s \mapsto \gamma(s) \in \mathbb{H} \) is called horizontal if its tangent vector \( \gamma'(s) \) is in the linear span of \( \{X_1(\gamma(s)), X_2(\gamma(s))\} \) for every \( s \) such that \( \gamma'(s) \) exists; in other words, there exist \( a(s), b(s) \in \mathbb{R} \) satisfying

\[
\gamma'(s) = a(s)X_1(\gamma(s)) + b(s)X_2(\gamma(s))
\]

whenever \( \gamma'(s) \) exists. We denote

\[
\|\gamma'(s)\| = \left( a^2(s) + b^2(s) \right)^{1/2}.
\]

Given \( p, q \in \mathbb{H} \), denote

\[
\Gamma(p, q) = \{ \text{horizontal curves } \gamma(s) \ (s \in [0, 1]): \gamma(0) = p \text{ and } \gamma(1) = q \}.
\]

Chow’s theorem states that \( \Gamma(p, q) \neq \emptyset \); see, for example, [2]. The Carnot-Carathéodory metric is then defined to be

\[
d_{CC}(p, q) = \inf_{\gamma \in \Gamma(p, q)} \int_0^1 \|\gamma'(s)\| \, ds.
\]

It is known that \( d_{CC} \) is equivalent to the Kórányi metric; see for example [6]. Another metric is

\[
d_R(p, q) = |p \cdot q^{-1}|_G.
\]

It is a right-invariant metric and therefore very different from \( d_L \) and \( d_{CC} \). Let us quickly mention below the relation between \( d_L \) and \( d_R \). It turns out that \( d_L \) and \( d_R \) are not bi-Lipschitz equivalent (cf. [21, Example 2.1]), but one is locally \( 1/2 \)-Hölder continuous with respect to the other, as described in the proposition below with its proof omitted here.
Proposition 2.1 (Hölder continuity of metric conversion, [21, Proposition 2.2]). For any \( \rho > 0 \), there exists \( C_\rho > 0 \) such that
\[
d_L(p, q) \leq C_\rho d_R(p, q)^{1/2}
\]
for any \( p, q \in \mathbb{H} \) with \( |p|, |q| \leq \rho \).

It is therefore quite clear that functions that are Lipschitz continuous with respect to either of the metrics is not necessarily Lipschitz continuous with respect to the other.

On the other hand, both types of Lipschitz continuity are equivalent under additional symmetry.

Proposition 2.2 (Equivalence of Lipschitz functions with symmetry, [21, Proposition 2.5]). Let \( f : \mathbb{H} \to \mathbb{R} \) be a function that is either symmetric about the origin or symmetric about the horizontal coordinate plane, i.e., \( f \) satisfies either
\[
f(p_1, p_2, p_3) = f(-p_1, -p_2, -p_3) \quad \text{or} \quad f(p_1, p_2, p_3) = f(-p_1, -p_2, p_3)
\]
for all \( (p_1, p_2, p_3) \in \mathbb{H} \). Then \( f \) is Lipschitz with respect to \( d_L \) if and only if it is Lipschitz with respect to \( d_R \).

2.2. Viscosity solutions in the Heisenberg group. For our later application of the Crandall-Ishii lemma (cf. [8]) to the Heisenberg group, let us recall the definition of semijets adapted to the Heisenberg group: for any \( (p, t) \in \mathbb{H} \times (0, \infty) \) and any locally bounded upper semicontinuous function \( u : \mathbb{H} \times (0, \infty) \to \mathbb{R} 

\[
P_{H}^{2+} u(p, t) = \left\{ (\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^2 : u(q, s) \leq u(p, t) + \tau(s - t)
\right. 
\]
\[
+ \langle \zeta, p^{-1} \cdot q \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|p^{-1} \cdot q|^2_G) \}
\]
where \( h \) denotes the horizontal projection of \( p^{-1} \cdot q \). Similarly, we may define
\[
P_{H}^{2-} u(p, t) = \left\{ (\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^2 : u(q, s) \geq u(p, t) + \tau(s - t)
\right. 
\]
\[
+ \langle \zeta, p^{-1} \cdot q \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|p^{-1} \cdot q|^2_G) \}
\]
for any locally bounded lower semicontinuous function \( u \). Also, the closure \( \overline{P}_{H}^{2+} \) is the set of triples \( (\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^2 \) that satisfy the following: there exist \( (p_j, t_j) \in \mathbb{H} \times [0, \infty) \) and \( (\tau_j, \zeta_j, X_j) \in \overline{P}_{H}^{2+}(p_j, t_j) \) such that
\[
(p_j, t_j, u(p_j, t_j), \tau_j, \zeta_j, X_j) \to (p, t, u(p, t), \tau, \zeta, X) \quad \text{as} \ j \to \infty.
\]
The closure set \( \overline{P}_{H}^{2-} \) of \( \overline{P}_{H}^{2-} \) can be similarly defined. We refer to [4] for more details.

We say \( u : \mathbb{H} \times [0, \infty) \to \mathbb{R}^m \) is an upper (resp., lower) semicontinuous vector-valued function if each component \( u_k \) (\( k=1, 2, \ldots, m \)) of \( u \) is upper (resp., lower) semicontinuous.

Definition 2.3. A bounded upper semicontinuous vector-valued function \( u : \mathbb{H} \times [0, \infty) \to \mathbb{R}^m \) is said to be a viscosity subsolution of (1.1) if
\[
\tau + F_i(p, u(p, t), \zeta, X) \leq 0
\]
for all \((p, t) \in \mathbb{H} \times (0, \infty)\), \((\tau, \zeta, X) \in P^{2+}_H u_i(p, t)\) and all \(i = 1, 2, \ldots, m\). Similarly, a bounded lower semicontinuous vector-valued function \(u : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{R}^m\) is said to be a viscosity supersolution of \((1.1)\) if
\[
\tau + F_i(p, u(p, t), \zeta, X) \geq 0 \tag{2.4}
\]
for all \((p, t) \in \mathbb{H} \times (0, \infty)\), \((\tau, \zeta, X) \in P^{-2}_H u_i(p, t)\) and all \(i = 1, 2, \ldots, m\). A bounded continuous function \(u : \mathbb{H} \times [0, \infty) \rightarrow \mathbb{R}^m\) is called a viscosity solution of \((1.1)\) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 2.4.** It is not difficult to see that a bounded upper semicontinuous (resp., lower semicontinuous) function \(u\) is a subsolution (resp., supersolution) if and only if \((2.3)\) (resp., \((2.4)\)) holds for all \((p, t) \in \mathbb{H} \times (0, \infty)\), \((\tau_i, \zeta)\) is an upper semicontinuous function and a lower semicontinuous function on \(H\), and any fixed \(i\), we may find \((p_j, t_j) \in \mathbb{H} \times (0, \infty)\) and \((\tau_j, \zeta_j, X_j) \in P^{2+}_H u_i(p_k, t_k)\) with
\[
(p_j, t_j, u(p_j, t_j), \tau_j, \zeta_j, X_j) \rightarrow (p, t, u(p, t), \tau, \zeta, X) \quad \text{as} \quad j \rightarrow \infty.
\]
Taking a subsequence if necessary, we get \(r' = (r'_1, r'_2, \ldots, r'_m) \in \mathbb{R}^m\) such that
\[
u(p_j, t_j) \rightarrow r' \quad \text{as} \quad j \rightarrow \infty.
\]
It is clear that \(r' \leq \nu(p, t)\) due to the upper semicontinuity of \(u\). Also, we have \(r'_i = u_i(p, t)\).

Since \((2.3)\) holds at \((p_j, t_j)\), i.e.,
\[
\tau_j + F_i(p_j, u(p_j, t_j), \zeta_j, X_j) \leq 0,
\]
by using the continuity of \(F_i\) we obtain
\[
\tau + F_i(p, r', \zeta, X) \leq 0,
\]
which in turn implies \((2.3)\) at \((p, t)\). Here we adopted the monotonicity assumption \((A2)\) of \(F_i\); in our current case, we take \(\tau = \nu(p, t)\), which yields
\[
\max_k (r_k - r'_k) = u_i(p, t) - r'_i = 0.
\]

**Remark 2.5.** By \((A2)\), it is clear that for any \(c \geq 0\) (resp., \(c \leq 0\)), \(u_c = (u_1 + c, u_2 + c, \ldots, u_m + c)\) is a supersolution (resp., subsolution) provided that \(u\) is a supersolution (resp., subsolution).

We next present an adapted version of the celebrated Crandall-Ishii lemma for single equations in the viscosity solution theory; see [22, 4] for the detailed proof. In the sequel, we denote by \(\nabla^H\) the horizontal gradient of a multivariable function with respect to the variable \(p \in \mathbb{H}\).

**Lemma 2.6** (Adaptation of the Crandall-Ishii lemma to the Heisenberg group). Let \(\phi\) be a smooth function in \(\mathbb{H} \times [0, T]\) for an arbitrary \(T > 0\). Suppose that \(u\) and \(v\) are resp. an upper semicontinuous function and a lower semicontinuous function on \(\mathbb{H} \times [0, T]\). If there exist \(p_0, q_0 \in \mathbb{H}, t_0, s_0 \in (0, T)\) such that
\[
\max_{p, q \in \mathbb{H}, t, s \in (0, T)} (u(p, t) - v(q, s) - \phi(p, q, t, s)) = u(p_0, t_0) - v(q_0, s_0) - \phi(p_0, q_0, t_0, s_0),
\]
then there exist \((\tau, \xi, X) \in \overline{P}^{2+}_H u(p_0, t_0)\) and \((\mu, \eta, Y) \in \overline{P}^{2-}_H v(q_0, s_0)\) such that
\[
\tau - \mu = \partial_t \phi(p_0, q_0, t_0, s_0) + \partial_s \phi(p_0, q_0, t_0, s_0),
\]
\[
\xi_h = \nabla^H_{\partial p} \phi(p_0, q_0, t_0, s_0), \quad \eta_h = -\nabla^H_{\partial q} \phi(p_0, q_0, t_0, s_0),
\]
and
\[ (Xw, w) - (Yw, w) \leq \langle \nabla^2 \phi(p_0, q_0, s_0)(w_{p_0} \oplus w_{q_0}), w_{p_0} \oplus w_{q_0} \rangle, \]
for all \( w = (w_1, w_2) \in \mathbb{R}^2 \), where \( w_p \in \mathbb{R}^3 \) is given by
\[ w_p = \left( w_1, w_2, \frac{1}{2} (x_p w_2 - y_p w_1) \right) \]
for every \( p = (x_p, y_p, z_p) \in \mathbb{H} \).

3. Uniqueness of Viscosity Solutions

**Theorem 3.1** (Comparison principle). Suppose that \( F_i (i = 1, 2, \ldots, m) \) satisfies (A1), (A2) and (A3). Let \( u \) and \( v \) be respectively a subsolution and a supersolution of (1.1). Assume that for any fixed \( T > 0 \), there exist \( k > 0 \) and \( C_T > 0 \) depending on \( T \) such that
\[ |u(p, t) - v(p, t)| \leq C_T e^{k(p)} \tag{3.1} \]
for all \( (p, t) \in \mathbb{H} \times [0, T] \), where
\[ \langle p \rangle = (1 + x^4 + y^4 + 16z^2)^{\frac{1}{4}} \quad \text{for all } p = (x, y, z) \in \mathbb{H}. \tag{3.2} \]
If \( u(\cdot, 0) \leq v(\cdot, 0) \) in \( \mathbb{H} \), then \( u \leq v \) in \( \mathbb{H} \times [0, \infty) \).

As an immediate consequence, we obtain the uniqueness of viscosity solutions, if they exist, to (1.1)–(1.2) that satisfy the following growth condition at space infinity:

**(G)** For any fixed \( T > 0 \), there exist \( k > 0 \) and \( C_T > 0 \) such that
\[ |u(p, t)| \leq C_T e^{k(p)} \]
for all \( (p, t) \in \mathbb{H} \times [0, T] \).

We provide the following elementary lemma, which will be useful later in the proof of Theorem 3.1.

**Lemma 3.2** (Elementary estimates for the penalty at space infinity). Let \( \langle p \rangle \) be defined as in (3.2). Then there exists \( C > 0 \) such that
\[ |\nabla H \langle p \rangle| + |(\nabla^2_H)^* \langle p \rangle| \leq C \quad \text{for all } p \in \mathbb{H}. \tag{3.3} \]

**Proof.** Set, for every \( p = (x, y, z) \in \mathbb{H} \),
\[ f(p) = x^4 + y^4 + 16z^2. \]
Then we can write
\[ \langle p \rangle^4 = 1 + f(p). \]
Differentiating this relation, we get
\[ 4 \langle p \rangle^3 \nabla_H \langle p \rangle = \nabla_H f(p) = (4x^3 - 16yz, 4y^3 + 16xz), \tag{3.4} \]
It follows that
\[ \nabla_H \langle p \rangle = \left( \frac{x^3 - 4yz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}}, \frac{y^3 + 4xz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}} \right), \tag{3.5} \]
which implies that
\[ |\nabla_H \langle p \rangle| \leq C_1 \]
for some $C_1 > 0$. We then further differentiate (3.4) and obtain

$$12 \langle p \rangle^2 \nabla_H (p) \otimes \nabla_H (p) + 4 \langle p \rangle^3 \nabla^2_H (p) = \nabla^2_H f(p).$$

(3.6)

Since we have by direct calculations

$$\frac{1}{4} \nabla^2_H f(p) = \begin{pmatrix} 3x^2 + 2y^2 & 4z - 2xy \\ -4z - 2xy & 3y^2 + 2x^2 \end{pmatrix},$$

and therefore $|((\nabla^2_H f)^*) (p)| \leq 4C_2 \langle p \rangle^2$ for some $C_2 > 0$, it follows from (3.6) that

$$|((\nabla^2_H f)^*) (p)| \leq (C_2 + 3C_1^2) (p)^{-1}$$

for all $p \in \mathbb{H}$. Hence, we obtain the desired inequality (3.3) for an appropriate bound $C > 0$.

\[ \square \]

**Proof of Theorem 3.1.** We assume by contradiction that $u_i - v_i$ takes a positive value at some $(p, t) \in \mathbb{H} \times (0, \infty)$ for some $i = 1, 2, \ldots, m$. By the growth assumption, there exist $k > 0$ and $C_T > 0$ satisfying (3.1). Take an arbitrary constant $\beta > \max \{k, 1\}$ and $\alpha > 0$ to be determined at the end of the proof. Recall the function $\langle p \rangle$ given in (3.2) and set

$$g(p, t) = e^{\alpha t + \beta \langle p \rangle}$$

(3.7)

for $(p, t) \in \mathbb{H} \times [0, \infty)$.

Note that by Lemma 3.2 we get a constant $C_\beta > 0$ such that

$$|\nabla_H g(p, t)| + |(\nabla^2_H g)^*(p, t)| \leq C_\beta g(p, t)$$

(3.8)

for all $(p, t) \in \mathbb{H} \times [0, \infty)$.

Then there exists $\sigma \in (0, 1)$ such that

$$\max_{i=1,\ldots,m} u_i(p_i, t_i) - v_i(p_i, t_i) - 2\sigma g(p_i, t_i) - \frac{\sigma}{T - t_i} = u_i(p, t) - v_i(p, t) - 2\sigma g(p, t) - \frac{\sigma}{T - t} > 0$$

for some $l \in \{1, 2, \ldots, m\}$ and $(p, t) \in \mathbb{H} \times [0, T)$. For all $\varepsilon > 0$ small, consider the function

$$\Phi_i(p, q, t, s) = u_i(p, t) - v_i(q, s) - \sigma \Psi_\varepsilon(p, q, t, s) - \frac{(t - s)^2}{\varepsilon} - \frac{\sigma}{T - t} - \frac{\sigma}{T - s}$$

with

$$\Psi_\varepsilon(p, q, t, s) = \varphi_\varepsilon(p, q) + K(p, q, t, s),$$

$$\varphi_\varepsilon(p, q) = \frac{1}{\varepsilon} d_R(p, q)^4 = \frac{|p - q|^4}{\varepsilon}, \quad K(p, q, t, s) = g(p, t) + g(q, s).$$

Then $\max \Phi_i$ attains a positive maximum at some $(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H}^2 \times [0, T)^2$. In particular, there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\Phi_j(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \Phi_i(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)$$

for any $i = 1, 2, \ldots, m$ and

$$\Phi_j(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \varphi_\varepsilon(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon).$$

The former yields

$$\max_i (u_i(p_\varepsilon, t_\varepsilon) - v_i(q_\varepsilon, s_\varepsilon)) = u_j(p_\varepsilon, t_\varepsilon) - v_j(q_\varepsilon, s_\varepsilon) \geq 0$$

(3.9)
while the latter implies that
\[
\frac{|p_\varepsilon \cdot q_\varepsilon^{-1}|}{\varepsilon} \left| \frac{(t_\varepsilon - s_\varepsilon)^2}{\varepsilon} \right| \leq u_j(p_\varepsilon, t_\varepsilon) - v_j(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon} - \frac{\sigma}{T - s_\varepsilon} - \left( u_l(\hat{p}, \hat{t}) - v_l(\hat{p}, \hat{t}) - 2\sigma g(\hat{p}, \hat{t}) - \frac{\sigma}{T - \hat{t}} \right). \tag{3.10}
\]
Since, due to (3.1) again, the quantity \( u_j(p_\varepsilon, t_\varepsilon) - v_j(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) \) is bounded from above uniformly in \( \varepsilon \), we have
\[
|p_\varepsilon \cdot q_\varepsilon^{-1}| \rightarrow 0 \quad \text{and} \quad t_\varepsilon - s_\varepsilon \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{3.11}
\]
We notice that \( p_\varepsilon, q_\varepsilon \) are bounded, since otherwise the right hand side of (3.10) will tend to \(-\infty\). Therefore, by taking a subsequence, still indexed by \( \varepsilon \), we have \( p_\varepsilon, q_\varepsilon \rightarrow \bar{p} \in \mathbb{H} \) and \( t_\varepsilon, s_\varepsilon \rightarrow \bar{t} \in [0, T) \). It follows that
\[
\limsup_{\varepsilon \to 0} u_j(p_\varepsilon, t_\varepsilon) - v_j(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon} - \frac{\sigma}{T - s_\varepsilon} \leq u_j(\bar{p}, \bar{t}) - v_j(\bar{p}, \bar{t}) - 2\sigma g(\bar{p}, \bar{t}) - \frac{2\sigma}{T - \bar{t}},
\]
which yields
\[
\varphi_\varepsilon(p_\varepsilon, q_\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]
Also, it is easily seen that \( \bar{t} > 0 \) and therefore \( t_\varepsilon, s_\varepsilon > 0 \) thanks to the condition that \( u(\cdot, 0) \leq v(\cdot, 0) \) in \( \mathbb{H} \).

Following [22], we now apply Lemma 2.6 and obtain, for any \( \lambda \in (0, 1) \) independent of \( \varepsilon \),
\[
(a_1, \zeta_1, X) \in \mathcal{P}_H^{2^+} u_j(p_\varepsilon, t_\varepsilon) \quad \text{and} \quad (a_2, \zeta_2, Y) \in \mathcal{P}_H^{2^-} v_j(q_\varepsilon, s_\varepsilon)
\]
such that
\[
a_1 - a_2 = \alpha \sigma K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) + \frac{\sigma}{(T - t_\varepsilon)^2} + \frac{\sigma}{(T - s_\varepsilon)^2} \geq \alpha \sigma K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) + \frac{2\sigma}{T^2}, \tag{3.12}
\]
\[
\langle Xw, w \rangle - \langle Yw, w \rangle \leq \langle (\sigma M + \lambda \sigma^2 M^2)w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \rangle, \tag{3.13}
\]
and the horizontal projections of \( \zeta_1, \zeta_2 \in \mathbb{R}^3 \) can be written respectively as \( \xi + \eta_1 \) and \( \xi + \eta_2 \) (in \( \mathbb{R}^2 \)) with
\[
\xi = \nabla^p_H \varphi_\varepsilon(p_\varepsilon, q_\varepsilon), \quad \eta_1 = \beta \sigma \nabla^p g(p_\varepsilon, t_\varepsilon), \quad \eta_2 = -\beta \sigma \nabla^q g(q_\varepsilon, s_\varepsilon).
\]
Here \( w = (w_1, w_2) \) is an arbitrary vector in \( \mathbb{R}^2 \), \( M = (\nabla^2 \varphi_\varepsilon)(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in S^6 \), and
\[
w_{p_\varepsilon} = \left( w_1, w_2, \frac{1}{2} w_2 x_{p_\varepsilon} - \frac{1}{2} w_1 y_{p_\varepsilon} \right) \tag{3.14}
\]
and
\[
w_{q_\varepsilon} = \left( w_1, w_2, \frac{1}{2} w_2 x_{q_\varepsilon} - \frac{1}{2} w_1 y_{q_\varepsilon} \right) \tag{3.15}
\]
with \( p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon}) \) and \( q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon}) \).

It is easily seen that \( M = M_1 + M_2 \), where
\[
M_1 = \nabla^2 \varphi_\varepsilon(p_\varepsilon, q_\varepsilon)
\]
and
\[
M_2 = \nabla^2 K(p_\varepsilon, q_\varepsilon) = \begin{pmatrix}
\nabla^2 g(p_\varepsilon, t_\varepsilon) & 0 \\
0 & \nabla^2 g(q_\varepsilon, s_\varepsilon)
\end{pmatrix}.
\]
Let us estimate the right hand side of (3.13), following the strategy in [21] for the case of single equations, which extends the Euclidean argument in [1] to the Heisenberg group.

It follows from the calculation in the comparison arguments in [4] (and also [3, 22, 21]) that there exists \(C > 0\) such that
\[
\langle (M_1 + \lambda \sigma M_1^2)w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \rangle \leq \frac{C}{\varepsilon} |w|^2 (z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon} y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon} y_{p_\varepsilon})^2 \\
\leq C \varphi_\varepsilon(p_\varepsilon, q_\varepsilon)|w|^2
\]
for \(\lambda > 0\) small.

By (3.8), we may let \(C_\beta > 0\) be larger if necessary such that
\[
\langle M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon}) \rangle = \langle \nabla_M^2 K(p_{\varepsilon}, q_{\varepsilon}, t_{\varepsilon}, s_{\varepsilon})w \oplus w, w \oplus w \rangle \\
= \langle \nabla_M^2 g(p_{\varepsilon}, t_{\varepsilon})w, w \rangle + \langle \nabla_M^2 g(q_{\varepsilon}, s_{\varepsilon})w, w \rangle \\
\leq |w|^2 C_\beta K(p_{\varepsilon}, q_{\varepsilon}, t_{\varepsilon}, s_{\varepsilon})
\]
for \(C_\beta > 0\) arbitrarily small. The right hand side carries the term \(|w|^2\) due to the fact that both \(w_{p_\varepsilon}\) and \(w_{q_\varepsilon}\), given respectively in (3.14) and (3.15), can be viewed as homogeneous functions of \(w\) with degree 1.

Employing the estimates (3.16), (3.17) and (3.18), and taking \(\lambda > 0\) sufficiently small in (3.13), we have
\[
\langle Xw, w \rangle - \langle Yw, w \rangle \leq C\sigma |w|^2 \varphi_\varepsilon(p_{\varepsilon}, q_{\varepsilon}) + \sigma |w|^2 C_\beta K(p_{\varepsilon}, q_{\varepsilon}, t_{\varepsilon}, s_{\varepsilon}) + c_\lambda |w|^2.
\]

We next apply the definition of viscosity sub- and supersolutions to the \(j\)-th equation of our weakly coupled system provided in Definition 2.3 and Remark 2.4. It turns out that
\[
a_1 + F_j(p_{\varepsilon}, r_1, r_2, \ldots, r_m, \xi + \eta_1, X) \leq 0
\]
and
\[
a_2 + F_j(q_{\varepsilon}, r'_1, r'_2, \ldots, r'_m, \xi + \eta_2, Y) \geq 0,
\]
where we write \(r_i = u_i(p_{\varepsilon}, t_{\varepsilon})\) and \(r'_i = v_i(q_{\varepsilon}, s_{\varepsilon})\) for all \(i = 1, 2, \ldots, m\).

By subtracting (3.21) from (3.20), we have
\[
a_1 - a_2 \leq I_1 + I_2 + I_3,
\]
where
\[
I_1 = F_j(q_{\varepsilon}, r'_1, r'_2, \ldots, r'_m, \xi + \eta_2, Y) - F_j(p_{\varepsilon}, r'_{1}, r'_{2}, \ldots, r'_{m}, \xi + \eta_1, Y),
\]
\[
I_2 = F_j(p_{\varepsilon}, r'_1, r'_2, \ldots, r'_m, \xi + \eta_1, Y) - F_j(p_{\varepsilon}, r'_{1}, r'_{2}, \ldots, r'_{m}, \xi + \eta_1, X),
\]
and
\[
I_3 = F_j(p_{\varepsilon}, r'_1, r'_2, \ldots, r'_m, \xi + \eta_1, X) - F_j(p_{\varepsilon}, r_1, r_2, \ldots, r_m, \xi + \eta_1, X).
\]

Due to the assumption (A1), by further taking the constant \(C_\beta\) larger, we have
\[
I_1 \leq L|p_{\varepsilon} \cdot q_{\varepsilon}^{-1}|_G + \sigma C_\beta L K(p_{\varepsilon}, q_{\varepsilon}, t_{\varepsilon}, s_{\varepsilon}).
\]
Also, adopting the parabolicity assumption (A3) together with (A1) and (3.19), we are led to
\[
I_2 \leq C\sigma L \varphi_\varepsilon(p_{\varepsilon}, q_{\varepsilon}) + \sigma C_\beta L K(p_{\varepsilon}, q_{\varepsilon}, t_{\varepsilon}, s_{\varepsilon}) + c_\lambda L.
\]
Moreover, (3.9) enables us to apply the monotonicity assumption (A2) and obtain
\[ I_3 \leq 0. \tag{3.25} \]

It follows from (3.22), (3.23), (3.24) and (3.25) that
\[ a_1 - a_2 \leq c_\lambda L + L \left( |p_\epsilon \cdot q_\epsilon^{-1}|_G + C \sigma \varphi_\epsilon(p_\epsilon, q_\epsilon) \right) + 2\sigma C_{1,3} K(p_\epsilon, q_\epsilon, t_\epsilon, s_\epsilon). \tag{3.26} \]

Note that by (3.12),
\[ a_1 - a_2 \geq \frac{\sigma}{T^2} + \alpha \sigma K(p_\epsilon, q_\epsilon, t_\epsilon, s_\epsilon). \]

Letting \( \lambda \to 0 \) and then \( \epsilon \to 0 \), we deduce
\[ 2\sigma \left( \frac{1}{T^2} + \frac{\alpha}{T^2} K(p_\epsilon, q_\epsilon, t_\epsilon, s_\epsilon) \right) \leq 2\sigma C_{1,3} K(p_\epsilon, q_\epsilon, t_\epsilon, s_\epsilon), \]
which is a contradiction if we take \( \alpha > 2C_{1,3} \).

4. Existence of Viscosity Solutions

Inspired by the deterministic game-theoretic interpretations of fully nonlinear partial differential equations in [19], we here provide an iterated scheme to approximate the unique solution of (1.1)–(1.2). In order to handle the unboundedness of initial condition at infinity, we incorporate the growth rate in the scheme.

Suppose that the initial value satisfies the growth condition (G0). Then we fix \( \alpha, \beta, \gamma > 0 \) satisfying
\[ 0 < \alpha < \frac{1}{3}, \quad \alpha + \beta < 1, \quad \alpha + \gamma < 1. \tag{4.1} \]

Set
\[ A = \{ \omega \in \mathbb{H}_0 : |\omega|_G \leq \epsilon^{-\alpha} \}, \]
\[ B_p = \{ \xi \in \mathbb{R}^2 : |\xi| \leq \epsilon^{-\beta} e^{k(p)} \}, \]
\[ C_p = \{ X \in S^2 : \|X\| \leq \epsilon^{-\gamma} e^{k(p)} \}. \]

Fix \((p, t) \in \mathbb{H} \times [0, \infty)\). Take \( \epsilon > 0 \) small. We inductively define a vector-valued function as follows: for every \( i = 1, 2, \ldots, m \),
\[ u_1^\epsilon(p, t) = \sup_{\xi \in B_p} \inf_{w \in A} \left\{ \frac{u_1^\epsilon(p \cdot \epsilon w, t - \epsilon^2) - \epsilon \langle \xi, \dot{w} \rangle - \epsilon^2}{2} (X \dot{w}, \dot{w}) - \frac{\epsilon^2}{2} \right\} \tag{4.2} \]

for \( t \geq \epsilon^2 \) and
\[ u_1^\epsilon(p, t) = h_i(p) \tag{4.3} \]

for \( t < \epsilon^2 \). Here \( \dot{w} \) denotes the horizontal projection of \( w \in \mathbb{H} \); that is, for any \( w = (w_1, w_2, w_3) \in \mathbb{H} \), we have \( \dot{w} = (w_1, w_2) \in \mathbb{R}^2 \). In other words, we need to run the scheme for \( \lceil t/\epsilon^2 \rceil \) steps to calculate \( u_1^\epsilon(p, t) \) for all \( p \in \mathbb{H}, t \geq 0 \).

Theorem 4.1 (Existence of solutions and convergence of scheme). Assume that (A1)–(A3) hold. Suppose that \( h = (h_1, h_2, \ldots, h_m) : \mathbb{H} \to \mathbb{R}^m \) is continuous and satisfies the growth condition (G0). Let \( u^\epsilon \) be given as in (4.2)–(4.3). Then \( u^\epsilon \to u \) locally uniformly in \( \mathbb{H} \times [0, \infty) \) as \( \epsilon \to 0 \), where \( u \) is a unique viscosity solution of (1.1)–(1.2).
We prove the theorem in the following subsections. In Section 4.1, we first verify convergence of the scheme to the nonlinear operator in the case of smooth functions. Since we allow solutions to grow exponentially at space infinity, our estimates always carry the growth, which is different from the bounded case [19]. Section 4.2 is devoted to showing that the scheme keeps the exponential growth in space. Based on the results obtained in Section 4.1 and Section 4.2, we give the proof of Theorem 4.1 in Section 4.3, by using the standard viscosity argument involving half relaxed limits.

4.1. Consistency of the scheme. For our convenience, for every \( p \in \mathbb{H} \) and \( i = 1, 2, \ldots, m \), define \( T_i^p : (C(\mathbb{H}))^m \rightarrow \mathbb{R} \) to be

\[
T_i^p(u) = \sup_{\xi \in B_p} \inf_{w \in A} \left\{ u_i(p, \varepsilon w) - \varepsilon \langle \xi, \tilde{w} \rangle - \frac{\varepsilon^2}{2} \langle X \tilde{w}, \tilde{w} \rangle - \varepsilon^2 F_i(p, u(p), \xi, X) \right\}.
\]

(4.4)

It is then easily seen, by (4.2), that

\[
u_i^\varepsilon(p, t) = T_i^p(u^\varepsilon(\cdot, t - \varepsilon^2))
\]

for any \( t \geq \varepsilon^2 \) and \( i = 1, 2, \ldots, m \).

For any \( i = 1, 2, \ldots, m \), we denote by \( C^\infty_{i,k}(\mathbb{H}) \) the collection of all locally bounded vector-valued functions \( \varphi = (\varphi_1, \ldots, \varphi_m) : \mathbb{H} \rightarrow \mathbb{R}^m \) such that \( \varphi_i \) is smooth in \( \mathbb{H} \) with

\[
c_{i,k}(\varphi) := \sup \left\{ \left\| \left( X_i^1 X_2^l X_3^{l_3} \varphi_i \right) e^{-k(\cdot)} \right\|_{L^\infty} : l_1, l_2, l_3 = 0, 1, 2, \ldots \right\} < \infty.
\]

(4.6)

We also denote \( C^\infty_k(\mathbb{H}) = \bigcap_{i=1}^m C^\infty_{i,k}(\mathbb{H}) \) and let

\[
c_k(\varphi) = \max_{i=1,2,\ldots,m} c_{i,k}(\varphi)
\]

for any \( \varphi \in C^\infty_k(\mathbb{H}) \).

The consistency lemma is an adaptation of [19, Lemma 4.1] to our sub-Riemannian setting. It plays a key role in the scheme convergence.

**Lemma 4.2** (Consistency of the scheme). Suppose that \( F \) satisfies (A1)–(A3) and (4.1) holds. For any \( i = 1, 2, \ldots, m \), let \( T_i^p \) be given as in (4.4). Let \( \varphi = (\varphi_1, \ldots, \varphi_m) \in C^\infty_{i,k}(\mathbb{H}) \) with \( c_{i,k}(\varphi) \) given as in (4.6). Then there exist \( \mu = \mu(\alpha, \beta, \gamma) \in (0, 1) \), \( C = C(L, m) > 0 \), \( \varepsilon_0 = \varepsilon_0(c_{i,k}(\varphi), C, \alpha, \beta, \gamma, \mu) \) such that

\[
|T_i^p(\varphi) - \varphi_i(p) + \varepsilon^2 F_i(p, \varphi(p), \nabla_{H} \varphi_i(p), (\nabla_{H}^2 \varphi_i)^*(p))| \leq C\varepsilon^{2+\mu} e^{k(p)}
\]

for any \( p \in \mathbb{H} \) when \( \varepsilon \leq \varepsilon_0 \).

**Proof.** To simplify our notation in the proof, we write \( c_{i,k} \) instead of \( c_{i,k}(\varphi) \) and denote

\[
F_i^\varphi = F_i(p, \varphi(p), \nabla_{H} \varphi_i(p), (\nabla_{H}^2 \varphi_i)^*(p)).
\]

We pick \( \mu > 0 \) to satisfy

\[
\mu < 1 - 3\alpha, \quad \mu < \alpha, \quad \mu + \gamma < 1 - \alpha.
\]

(4.8)

For any \( \varphi \in C^\infty_{i,k}(\mathbb{H}) \), by Taylor’s formula in the Heisenberg group (cf. [22]), we get

\[
T_i^p(\varphi) - \varphi_i(p) = \sup_{\xi \in B_p} \inf_{w \in A} \left\{ \varepsilon \langle \nabla_{H} \varphi_i(p) - \xi, \tilde{w} \rangle + \frac{\varepsilon^2}{2} \langle (\nabla_{H}^2 \varphi_i)^*(p) - X \rangle \tilde{w}, \tilde{w} \rangle \right\} - \varepsilon^2 F_i(p, \varphi_i(p), \xi, X) + O(\varepsilon^3 e^{k(p)} |\tilde{w}|^3),
\]

(4.9)
where the constant implied in the error is clearly a multiple of \( c_{i,k} > 0 \). Then by comparing with the choices \( \xi = \nabla_H \varphi_i(p) \) and \( X = (\nabla_H^2 \varphi_i)^*(p) \), we can easily see that there is \( C > 0 \) such that
\[
T_i^p(\varphi) - \varphi(p) + \varepsilon^2 F_i^\varphi \geq -c_k C \varepsilon^{3-3\alpha} e^{k(p)} \geq -C \varepsilon^{2+\mu} e^{k(p)}
\]
when
\[
\varepsilon^{1-3\alpha-\mu} \leq \frac{1}{c_{i,k}}.
\]

It therefore suffices to show the inequality
\[
T_i^p(\varphi) - \varphi(p) + \varepsilon^2 F_i^\varphi \leq C \varepsilon^{2+\mu} e^{k(p)} \tag{4.10}
\]
for some \( C > 0 \). Following the strategy in [19], we divide our discussion into three cases depending on where the supremum over \( (\xi, X) \) is attained.

Case 1. Suppose that the maximizer \( (\xi, X) \) satisfies
\[
|\nabla_H \varphi_i(p) - \xi| \leq \varepsilon^\mu e^{k(p)} \quad \text{and} \quad \lambda_{\min} (\nabla_H^2 \varphi_i)^*(p) - X \geq -\varepsilon^\alpha e^{k(p)},
\]
where \( \lambda_{\min}(Y) \) denotes the least eigenvalue of \( Y \in S^2 \).

Then choosing \( \hat{w} = 0 \) (so that \( w = 0 \)), noticing
\[
(\nabla_H^2 \varphi_i)^*(p) + \varepsilon^\alpha e^{k(p)} I \geq X
\]
and using the parabolicity (A3) in (4.9), we obtain
\[
T_i^p(\varphi) - \varphi(p) \leq -\varepsilon^2 F_i \left( p, \varphi(p), \xi, (\nabla_H^2 \varphi_i)^*(p) + \varepsilon^{2+\mu} e^{k(p)} I \right).
\]

It follows from the Lipschitz continuity (A1) that
\[
T_i^p(\varphi) - \varphi(p) \leq -\varepsilon^2 F_i^\varphi + L \varepsilon^2 |\nabla_H \varphi_i(p) - \xi| + Lm \varepsilon^{2+\alpha} e^{k(p)}
\leq -\varepsilon^2 F_i^\varphi + C (\varepsilon^{2+\mu} + \varepsilon^{2+\alpha}) e^{k(p)},
\]
which immediately yields (4.10) due to the choice of \( \mu \) as in (4.8).

Case 2. Suppose that the maximizer \( (\xi, X) \) satisfies
\[
|\nabla_H \varphi_i(p) - \xi| \leq \varepsilon^\mu e^{k(p)} \quad \text{and} \quad \lambda_{\min} (\nabla_H^2 \varphi_i)^*(p) - X \leq -\varepsilon^\alpha e^{k(p)}.
\]

We then take \( \hat{w} \) with \( |\hat{w}| = \varepsilon^{-\alpha} \) to be an eigenvector with respect to the smallest eigenvalue
\[
\lambda = \lambda_{\min} (\nabla_H^2 \varphi_i)^*(p) - X.
\]
and also satisfy
\[
\langle \nabla_H \varphi_i - \xi, \hat{w} \rangle \leq 0.
\]

We again use (A1) to get the estimate
\[
T_i^p(\varphi) - \varphi(p) + \varepsilon^2 F_i^\varphi \leq \frac{1}{2} \varepsilon^{2-2\alpha} \lambda e^{k(p)} + C \varepsilon^{2+\mu} e^{k(p)} + C \varepsilon^2 |\lambda| e^{k(p)} + c_{i,k} C \varepsilon^{3-3\alpha} e^{k(p)}
\leq \frac{1}{2} \varepsilon^{2-2\alpha} |\lambda| e^{k(p)} + C \varepsilon^2 |\lambda| e^{k(p)} + C \varepsilon^{2+\mu} e^{k(p)} + c_{i,k} C \varepsilon^{3-3\alpha} e^{k(p)}.
\]

Since the first term on the right hand side is negative and dominant, we are led to (4.10) if \( \varepsilon > 0 \) further satisfies
\[
\varepsilon^{2\alpha} < \frac{1}{2C}.
\]

Case 3. Suppose that the maximizer \( (\xi, X) \) satisfies
\[
|\nabla_H \varphi_i(p) - \xi| \geq \varepsilon^\mu e^{k(p)}.
\]
This time we take \( w \in \mathbb{H}_0 \) with \( |\dot{w}| = \varepsilon^{-\alpha} \) such that
\[
\varepsilon \langle \nabla_H \varphi_i(p) - \xi, \dot{w} \rangle = -\varepsilon |\nabla_H \varphi_i(p) - \xi| |\dot{w}| \leq -\varepsilon^{1-\alpha+\mu} e^{k(p)}.
\]
Note that, if we additionally demand
\[
\varepsilon^{-\beta}, \varepsilon^{-\gamma} \geq c_{i,k},
\]
then we have
\[
\frac{\varepsilon^2}{2} \langle (\nabla_H^2 \varphi_i)^*(p) - X \rangle \dot{w}, \dot{w} \rangle \leq C \varepsilon^{2-\gamma-2\alpha} e^{k(p)}
\]
and, thanks to (A1) once again,
\[
\varepsilon^2 F_i^p - \varepsilon^2 F_i(p, \varphi(p), \xi, X) \leq C \varepsilon^{2-\beta} e^{k(p)} + C \varepsilon^{2-\gamma} e^{k(p)}.
\]
We apply the above estimates to (4.9) and obtain
\[
T_i^p(\varphi) - \varphi(p) + \varepsilon^2 F_i^p \leq (-\varepsilon^{1-\alpha+\mu} + C \varepsilon^{2-\gamma-2\alpha} + C \varepsilon^{2-\beta} + C \varepsilon^{2-\gamma} + C \varepsilon^{2+\mu}) e^{k(p)}.
\]
Since \( \mu \) satisfies \( \mu + \gamma < 1 - \alpha \), we have
\[
1 - \alpha + \mu < 2 - \gamma - 2\alpha < \min\{2 - \beta, 2 - \gamma\},
\]
which implies the first term in the parentheses is dominant. Hence, we end up with (4.10) again by letting
\[
\varepsilon^{1-\alpha-\mu-\gamma} \leq \frac{1}{C}.
\]

We finally collect all of the requirements on the smallness of \( \varepsilon \). It suffices to take \( \varepsilon \leq \varepsilon_0 \) with \( \varepsilon_0 \) satisfying
\[
\varepsilon_0^{2\alpha} + \varepsilon_0^{1-\alpha-\mu-\gamma} \leq C, \quad \varepsilon_0^{1-3\alpha-\mu} + \varepsilon_0^\beta + \varepsilon_0^\gamma \leq \frac{1}{c_{i,k}}.
\]
for some small \( C > 0 \) depending only on \( L \) and \( m \).

4.2. Growth at space infinity. We next discuss the growth of \( u^\varepsilon \) at space infinity.

**Proposition 4.3** (Growth at space infinity). Assume that (A1) holds. Let \( u^\varepsilon \) be given as in (4.2)–(4.3) with \( h \) satisfying (G0). Then for any \( T > 0 \), there exists \( C > 0 \) such that
\[
|u^\varepsilon(p, t)| \leq C e^{CT} e^{k(p)}
\]
for all \( (p, t) \in \mathbb{H} \times [0, T] \) and all \( \varepsilon > 0 \) sufficiently small.

**Proof.** We prove first that \( u_i^\varepsilon(p, t) \leq C e^{CT} e^{k(p)} \) for all \( i = 1, 2, \ldots, m \) for all \( t \geq 0 \). Let us provide an estimate for \( u^\varepsilon(\cdot, \varepsilon^2) \). In order to make comparisons, we set \( W : \mathbb{H} \to \mathbb{R} \) to be
\[
W(p) = C_0 e^{k(p)}
\]
and \( \mathbf{W} : \mathbb{H} \to \mathbb{R}^m \) to be
\[
\mathbf{W}(p) = (W(p), W(p), \ldots, W(p)).
\]
It is not difficult to see that \( \mathbf{W} \in C^\infty_\varepsilon(\mathbb{H}) \) with some constant \( C_k = c_k(\mathbf{W}) \) as defined in (4.7). We may assume that \( C_k > 1 \) for otherwise we just replace it by \( \max\{C_k, 1\} \) in the estimates below.

Fix \( i = 1, 2, \ldots, m \) arbitrarily. Adopting the Lipschitz continuity of \( F_i \) assumed in (A1), we have
\[
F_i(p, h(p), \xi, X) \geq F_i(p, \mathbf{W}(p), \xi, X) - C_k C e^{k(p)}
\]
(4.12)
for some $C > 0$. By the expression in (4.5), we get
\[
\begin{align*}
u_i^\varepsilon(p, \varepsilon^2) &= T^p_i(h) = \sup_{\xi \in B_p} \inf_{u \in A} \left\{ h_i(p \cdot \varepsilon w) - \varepsilon \langle \xi, \hat{w} \rangle - \frac{\varepsilon^2}{2} \langle X \hat{w}, \hat{w} \rangle - \varepsilon^2 F_i(p, \xi, \bar{X}) \right\},
\end{align*}
\]
which, in view of (4.12), yields
\[
T^p_i(h) \leq T^p_i(W) + \varepsilon^2 C_k e^{k(p)}.
\]
Applying Lemma 4.2, we obtain that
\[
u_i^\varepsilon(p, \varepsilon^2) - W(p) \leq T^p_i(W) - W(p) + \varepsilon^2 C_k e^{k(p)}
\leq -\varepsilon^2 F(p, W(p), \nabla H W(p), (\nabla^2 H W)^*(p)) + \varepsilon^2 C_k e^{k(p)} + C \varepsilon^{2+\mu} e^{k(p)}.
\]
By using Lemma 3.2 and (A1) again, we are led to
\[
u_i^\varepsilon(p, \varepsilon^2) - W(p) \leq \varepsilon^2 (1 + C_k C + C \varepsilon^\mu) e^{k(p)}.
\]
Let $\varepsilon$ be small such that $\varepsilon^\mu \leq 1$. It follows that
\[
u_i^\varepsilon(p, \varepsilon^2) \leq W(p) + \varepsilon^2 C_k e^{k(p)} \leq C_k (1 + \varepsilon^2) e^{k(p)}.
\]
We intend to iterate this estimate for all time $t \geq 0$. However, the smallness of $\varepsilon$, which the process heavily relies on, changes in each step. In order to have a uniform $\varepsilon$, we choose $\varepsilon_0$ that satisfies $\varepsilon_0^\mu \leq 1$ and also (4.11) with $c_k = 2C_k e^{CT}$.

Once we fix such an $\varepsilon_0$, we get
\[
C_k (1 + \varepsilon^2) \leq C_k (1 + \varepsilon^2)^N \leq 2C_k e^{CT},
\]
where $N = [T/\varepsilon^2]$ for all $\varepsilon \in (0, \varepsilon_0]$. Starting from (4.13) and repeating the same calculations as above, we have
\[
u_i^\varepsilon(p, 2\varepsilon^2) \leq C_k (1 + C \varepsilon^2)^2 e^{k(p)},
\]
where $C_k (1 + \varepsilon^2)^2 \leq 2C_k e^{CT}$ still holds. Continuing the iterations, we obtain
\[
u_i^\varepsilon(p, t) \leq 2C_k e^{CT} e^{k(p)}
\]
for all $(p, t) \times \mathbb{H} \times [0, T]$ and $i = 1, 2, \ldots, m$, as desired. We conclude the proof by observing that a symmetric argument can be used to show
\[
u_i^\varepsilon(p, t) \geq -2C_k e^{CT} e^{k(p)}
\]
for all $(p, t) \times \mathbb{H} \times [0, T]$ and $i = 1, 2, \ldots, m$. □

### 4.3. Proof of the existence theorem.
As is standard in the viscosity solution theory, we next define the half relaxed limits of $u^\varepsilon$. Let $\bar{u}, \underline{u} : \mathbb{H} \times [0, \infty) \to \mathbb{R}^m$ be vector-valued functions with
\[
\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m), \quad \underline{u} = (\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m),
\]
where
\[
\bar{u}_i = \limsup_{\varepsilon \to 0}^* u_i^\varepsilon, \quad \underline{u}_i = \liminf_{\varepsilon \to 0} u_i^\varepsilon.
\]
for all $i = 1, 2, \ldots, m$. We refer to [8] for more details about the definition and properties.

#### Proposition 4.4 (Relaxed limits as sub- and supersolutions).
Suppose that the components $u_i^\varepsilon$ of $u^\varepsilon : \mathbb{H} \times [0, \infty) \to \mathbb{R}^m$ are defined as in (4.2) and (4.3) for $i = 1, 2, \ldots, m$. Let $\bar{u}$ and $\underline{u}$ be given as in (4.14)–(4.15) respectively. Then $\bar{u}$ (resp., $\underline{u}$) is a viscosity subsolution (resp., supersolution) of (1.1).
We only show that\(^4\) to obtain an estimate of\(^5\) be given as in\(^6\)\(^7\)

for all\((p, t) \in \mathbb{H} \times [0, \infty)\) with\((p, t) \neq (p_0, t_0)\). By properly modifying values of\(\varphi\) far way from\((p_0, t_0)\), we may construct vector-valued functions\(\varphi^j \in C^\infty_{i, k}\) such that\(\varphi\) is the\(i\)-th component of\(\varphi^j\) and\(u^j_i\) is its\(j\)-th component for\(j \neq i\).

By taking a sequence, still indexed by\(\varepsilon\) for simplicity, we can find \((p_\varepsilon, t_\varepsilon) \in \mathbb{H} \times (0, \infty)\) such that

\[
(p_\varepsilon, t_\varepsilon) \to (p_0, t_0), \quad u^\varepsilon(p_\varepsilon, t_\varepsilon) \to r_u \quad \text{as} \quad \varepsilon \to 0
\]

for a certain\(r_u \in \mathbb{R}^m\) satisfying \(r_u \leq \underline{u}(p_0, t_0)\), and

\[
(u^\varepsilon_i - \varphi)(p_\varepsilon, t_\varepsilon) \geq (u^\varepsilon_i - \varphi)(p, t) - \varepsilon^4
\]

(4.16)

for any\((p, t) \in \mathbb{H} \times (0, \infty)\) satisfying \(|p^{-1} \cdot p_0| + |t - t_0| \leq \tau\) for some fixed\(\tau > 0\). Combining (4.2) and (4.16), we have

\[
\varphi(p_\varepsilon, t_\varepsilon) \leq T_i^p(\varphi^\varepsilon(\cdot, t - \varepsilon^2)) + \varepsilon^4.
\]

In view of Lemma 4.2, we get

\[
\begin{align*}
\varphi(p_\varepsilon, t_\varepsilon) - \varphi(p_\varepsilon, t_\varepsilon - \varepsilon^2) & \\
& \leq -\varepsilon^2 F_i(p_\varepsilon, \varphi(p_\varepsilon, t_\varepsilon - \varepsilon^2), \nabla_H \varphi_i(p_\varepsilon, t_\varepsilon^2 - \varepsilon^2), (\nabla_H^2 \varphi)^*(p_\varepsilon, t_\varepsilon - \varepsilon^2)) + o(\varepsilon^2)
\end{align*}
\]

for\(\varepsilon > 0\) small. Dividing the relation above by\(\varepsilon^2\) and sending\(\varepsilon \to 0\), we are led to

\[
\partial t \varphi(p_0, t_0) + F_i(p_0, r_u, \nabla_H \varphi_i(p_0, t_0), (\nabla_H^2 \varphi)^*(p_0, t_0)) \leq 0,
\]

which implies by (A2) that

\[
\partial t \varphi(p_0, t_0) + F_i(p_0, \underline{u}(p_0, t_0), \nabla_H \varphi_i(p_0, t_0), (\nabla_H^2 \varphi)^*(p_0, t_0)) \leq 0,
\]

as desired. \hfill \Box

**Proposition 4.5** (Verification of initial conditions). Suppose that \(u^j_i : \mathbb{H} \times [0, \infty) \to \mathbb{R}\) \((i = 1, 2, \ldots, m)\) are defined as in (4.2) and (4.3). Let \(\underline{u}\) and \(u\) be given as in (4.14)–(4.15) respectively. Then

\[
\underline{u}(\cdot, 0) \leq h \leq u(\cdot, 0) \quad \text{in} \quad \mathbb{H}.
\]

**Proof.** Fix an arbitrary \(p_0 \in \mathbb{H}\). Since \(h\) is continuous in \(\mathbb{H}\) with growth (G0), for any\(\delta > 0\), there exists \(L_\delta > 0\) such that \(h_i \leq W^j_i(\delta)\) in \(\mathbb{H}\) for all \(i = 1, 2, \ldots, m\), where \(W^j_i : \mathbb{H} \to \mathbb{R}\) is a smooth function given by

\[
W^j_i(p) = h_i(p_0) + \delta + \frac{L_\delta |p_0^{-1} \cdot p|^L_\delta}{1 + |p_0^{-1} \cdot p|^L_\delta} e^{k(p)}.
\]

It is not difficult to verify that \(W^i_\delta = (W^j_1, \ldots, W^m_\delta) \in C^\infty_k(\mathbb{H})\). We apply an analogous argument as in the proof of Proposition 4.3 to obtain an estimate of \(u^\varepsilon(\cdot, \varepsilon^2)\) from above. In fact, for all \(i = 1, 2, \ldots, m\),

\[
T^j_i(W^j_\delta) \leq W^j_\delta(p) + \varepsilon^2 CC_k(1 + C \varepsilon^2) e^{k(p)},
\]

where \(C_k \geq 1\) denotes an upper bound of the weighted norm \(c_k(W_\delta)\) while \(C > 0\) depends only on \(\delta, L_\delta, m, k\). Hence, utilizing Lemma 4.2 and (A1), we have

\[
u^j_i(p, \varepsilon^2) \leq W^j_\delta(p) + \varepsilon^2 CC_k e^{k(p)} := V^j_\delta(p)
\]
It is not difficult to see that the norm of the function with components $V^i_\delta(p)$ is bounded by $C_k(1 + C\varepsilon^2)$. We next use Lemma 4.2 on $V^i_\delta$ and obtain
\[ u^i_\delta(p, 2\varepsilon^2) \leq V^i_\delta(p) + \varepsilon^2CC_k(1 + C\varepsilon^2)e^{k(p)} \leq W^i_\delta(p) + 2C\varepsilon^2C_k(1 + C\varepsilon^2)e^{k(p)}. \]
Repeated applications of Lemma 4.2 and (A1) yield
\[ u^i(p, t) \leq W^i(p) + Cte^{C't}e^{k(p)} \]
for some $C > 0$ depending on $m, k, \delta, L_\delta$ and for all $i = 1, 2, \ldots, m$. It therefore gives
\[ \bar{u}(p_0, 0) \leq h(p_0) + \delta. \]
Letting $\delta \to 0$, we get
\[ \bar{u}(p_0, 0) \leq h(p_0). \]
The other half of (4.17) can be shown in a similar manner. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.4 and Proposition 4.3, $\bar{u}$ and $u$ are respectively a subsolution and a supersolution of (1.1) satisfying the exponential growth at space infinity. Since (4.17) holds due to Proposition 4.5, we can use the comparison principle, Theorem 3.1, to obtain
\[ \bar{u} \leq u \text{ in } \mathbb{H} \times [0, T) \]
for any $T > 0$. Then the results of the theorem follow immediately. □

5. Lipschitz Preserving Properties

We remark that the left-invariant Lipschitz preserving property for (1.1)–(1.2) does not hold in general. Even in the case of a single equation (i.e., $m = 1$ in our current setting), we have the following counterexample; see also [21, Example 4.1].

Example 5.1. Fix $h_0 = (1, 1) \in \mathbb{R}^2$. Let us consider the equation
\[ u_t - \langle h_0, \nabla_H u \rangle = 0 \text{ in } \mathbb{H} \]
with $u(p, 0) = u_0(p) = |p|_G$ for $p \in \mathbb{H}$. By direct verification, the unique solution is
\[ u(p, t) = |p \cdot ht|_G = d_R(p, h^{-1}t), \]
where $h = (1, 1, 0) \in \mathbb{H}_0$. However, it is not Lipschitz continuous with respect to $d_L$. One may choose $p_1 = (-t - \varepsilon, -t + \varepsilon, -\varepsilon t)$ and $p_2 = th^{-1} = (-t, -t, 0)$, which gives
\[ u(p_1, t) - u(p_2, t) = |p_1 \cdot p_2^{-1}|_G = (4\varepsilon^4 + 64\varepsilon^2 t^2)^{1/4} \]
but
\[ d_L(p_1, p_2) = |p_1^{-1} \cdot p_2|_G = \sqrt{2}\varepsilon. \]

On the other hand, there are also many examples where parabolic equations do preserve the left-invariant Lipschitz continuity, as shown in [21], where sufficient conditions for such a property to hold in the case of a single equation are given as well.

In what follows, we extend the results in [21] to the fully nonlinear system (1.1)–(1.2) by showing a right-invariant Lipschitz preserving property for the second-order system and then discussing several special cases.
5.1. Second-order parabolic systems.

**Theorem 5.2** (Preservation of right invariant Lipschitz continuity). **Assume that** $F_i$ satisfies (A1), (A2) and (A3). Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of (1.1)–(1.2) with $u_0$ Lipschitz continuous with respect to the right invariant metric $d_R$, i.e., there exists $L_0 > 0$ such that

$$|h(p) - h(q)| \leq L_0 d_R(p, q) \quad (5.1)$$

for all $p, q \in \mathbb{H}$. Then the solution $u$ is also Lipschitz in space with respect to $d_R$; more precisely, for any $t \geq 0$, there exists $L_1 > 0$ such that

$$|u(p, t) - u(q, t)| \leq (L_0 + L_1 t) d_R(p, q)$$

for all $p, q \in \mathbb{H}$.

**Proof.** We first remark that the uniqueness and existence of solutions $u$ are guaranteed by Theorem 3.1 and Theorem 4.1, since any right-invariant Lipschitz continuous function $h$ satisfies the growth condition (G0).

By symmetry, we only need to prove that for some $L_1 > 0$ large

$$u(p, t) \leq u(\omega^{-1} \cdot p, t) + (L_0 + L_1 t) |\omega|_G$$

(5.2)

for all $p, \omega \in \mathbb{H}$ and $t \geq 0$. To this end, we recall the left invariance of horizontal derivatives in the Heisenberg group, which implies that for any $\omega \in \mathbb{H}$,

$$u^{\omega}(p, t) = u(\omega^{-1} \cdot p, t)$$

is a solution of the system

$$\partial_t u_i + F_i(\omega^{-1} \cdot p, u, \nabla_H u_i, (\nabla^2_H)^* u_i) = 0$$

in $\mathbb{H} \times (0, \infty)$ for $i = 1, 2, \ldots, m$. It follows from (A1) that $u^{\omega}$ is a supersolution of

$$\partial_t u_i + F_i(p, u, \nabla_H u_i, (\nabla^2_H)^* u_i) = -L_1 |\omega|_G,$$

which yields that

$$v(p, t) = u(\omega^{-1} \cdot p, t) + (L_0 + L_1 t) |\omega|_G$$

is a supersolution of (1.1). Since

$$u(p, 0) \leq v(p, 0)$$

due to the Lipschitz continuity of the initial data, we apply Theorem 3.1 and obtain

$$u(p, t) \leq v(p, t),$$

which gives the desired inequality (5.2).

**Remark 5.3.** As an immediate consequence of Theorem 5.2 and Proposition 2.1, for any $\rho$, we obtain $C_\rho > 0$ such that

$$|u(p, t) - u(q, t)| \leq C_\rho d_L(p, q)^{3/2}$$

for all $p, q \in \mathbb{H}$ with $|p|_G, |q|_G \leq \rho$ and for all $t \geq 0$, if the initial condition $h$ is Lipschitz continuous in the sense of (5.1).

**Remark 5.4.** In view of Proposition 2.2, it follows from Theorem 5.2 that (1.1)–(1.2) does preserve the left-invariant Lipschitz continuity if the solution $u$ is known to be (component-wise) symmetric about the origin or the horizontal coordinate plane in $\mathbb{H}$. 
5.2. First-order Hamilton-Jacobi systems. Although the left-invariant Lipschitz preserving property for general second-order equations cannot be expected, it is however possible to obtain this property in a special case, where the operators are of first order and depend on \( u \) and the norm of its horizontal gradient. Namely, we are particularly interested in the following weakly coupled system of Hamilton-Jacobi equations.

\[
\begin{align*}
(HJ) \quad &\left\{ \begin{array}{ll}
\partial_t u_i + F_i(u_1, \ldots, u_m, |\nabla_H u_i|) = 0 & \text{in } \mathbb{H} \times (0, \infty) \text{ for } i = 1, \ldots, m, \\
u_i(\cdot, 0) = h_i & \text{in } \mathbb{H} \text{ for } i = 1, \ldots, m,
\end{array} \right.
\end{align*}
\]

If we additionally assume that the initial value is bounded in \( \mathbb{H} \), we obtain the following Lipschitz preserving result.

**Theorem 5.5** (Lipschitz preserving for Hamilton-Jacobi system). Suppose that \( F_i \) is independent of \( p \) and \( X \) and satisfies (A1) and (A2) for all \( i = 1, 2, \ldots, m \). Let \( u \) be the unique continuous viscosity solution of (HJ) with initial data \( h \) bounded and continuous in \( \mathbb{H} \). If \( h \) is Lipschitz with respect to \( d_L \) in \( \mathbb{H} \), i.e., there exists \( L > 0 \) such that (5.1) holds for any \( p, q \in \mathbb{H} \), then for all \( t \geq 0 \)

\[
|u(p, t) - u(q, t)| \leq Ld_L(p, q)
\]

for all \( p, q \in \mathbb{H} \).

**Proof.** Under the assumptions above, there is a unique continuous viscosity solution that is bounded in \( \mathbb{H} \times [0, T) \) for any fixed \( T > 0 \). We only need to show that

\[
u_i(p, t) - u_i(q, t) \leq Ld_L(p, q)
\]

for any \( p, q \in \mathbb{H} \), any \( t \in [0, T) \) and any \( i = 1, 2, \ldots, m \). The other part can be shown by a symmetric argument.

By Young’s inequality applied to (5.1), we obtain

\[
h_k(p) - h_k(q) \leq \frac{Ld_L(p, q)^4}{4\delta^4} + \frac{3L\delta^4}{4}
\]

for any \( \delta > 0 \), \( p, q \in \mathbb{H} \) and any \( k = 1, 2, \ldots, m \). It then suffices to show that

\[
u_k(p, t) - u_k(q, t) \leq \frac{Ld_L(p, q)^4}{4\delta^4} + \frac{3L\delta^4}{4}
\]

for all \( \delta > 0 \), \( p, q \in \mathbb{H} \) and any \( k = 1, 2, \ldots, m \). To this end, we fix \( \delta > 0 \) and define \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m) : \mathbb{H} \times [0, \infty) \to \mathbb{R} \) to be

\[
\tilde{u}_k(p, t) = \inf_{q \in \mathbb{H}} \{ u_k(q, t) + Cd_L(p, q)^4 \}
\]

with \( C = L/4\delta^4 \) for any \( k = 1, 2, \ldots, m \). It is clear that \( \tilde{u}_k \leq u_k \) in \( \mathbb{H} \times [0, \infty) \) for all \( k \).

We aim to show that \( \tilde{u} \) is a supersolution of (5.3). Note that \( u \) is bounded in \( \mathbb{H} \times [0, \infty) \) due to the boundedness of \( h \). This implies that for any \( (p, t) \in \mathbb{H} \times [0, \infty) \) the infimum in (5.6) can actually be attained at some \( q \in \mathbb{H} \). It is also clear that \( \tilde{u} \) is bounded in \( \mathbb{H} \times [0, \infty) \).

Suppose there exist a bounded open set \( \mathcal{O} \subset \mathbb{H} \times (0, T) \), \( \phi \in C^2(\mathcal{O}) \) and \( (\hat{p}, \hat{t}) \in \mathcal{O} \) such that \( \tilde{u}_i - \phi \) attains a unique maximum in \( \mathcal{O} \) at \( (\hat{p}, \hat{t}) \) for some \( i = 1, 2, \ldots, m \). We may also assume that \( \phi(p, t) \to -\infty \) when \( (p, t) \to \partial\mathcal{O} \). Then for any \( \epsilon > 0 \) sufficiently small,

\[
(p, t, s) \mapsto \tilde{u}_i(p, t) - \phi(p, s) + \frac{(t - s)^2}{\epsilon}
\]
attains a local minimum at some \((p_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H} \times [0, T) \times [0, T)\). Let \(q_\varepsilon\) be a minimizer in (5.6) for \((p, t) = (p_\varepsilon, t_\varepsilon)\). It then follows that
\[
\Phi_i(p, q, t, s) = u_i(q, t) + C d_L(p, q)^4 - \phi(p, s) + \frac{(t - s)^2}{\varepsilon}
\]
attains a local minimum at \((p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)\) for some \(q_\varepsilon \in \mathbb{H}\). Since \((\hat{p}, \hat{t})\) is the unique minimizer of \(\tilde{u}_i - \phi\), it is easily seen that \(p_\varepsilon \to \hat{p}\) and \(t_\varepsilon, s_\varepsilon \to \hat{t}\) as \(\varepsilon \to 0\), which, in particular, implies that \(t_\varepsilon, s_\varepsilon \neq 0\) for \(\varepsilon > 0\) small.

Moreover, by taking a subsequence if necessary, we get \(q_\varepsilon \to \hat{q}\) for a certain \(\hat{q} \in \mathbb{H}\). By continuity of \(u\), we have
\[
\tilde{u}_i(\hat{p}, \hat{t}) = u_i(\hat{q}, \hat{t}) + C d_L(\hat{p}, \hat{q})^4
\] (5.7)
and
\[
\tilde{u}_k(\hat{p}, \hat{t}) \leq u_k(\hat{q}, \hat{t}) + C d_L(\hat{p}, \hat{q})^4 \quad \text{for} \ k \neq i.
\] (5.8)
The minimum of \(\Phi_i\) also implies that
\[
\nabla_H \phi_1(p_\varepsilon) = \nabla_H \phi(p_\varepsilon, s_\varepsilon) \quad \text{and} \quad \phi_i(p_\varepsilon, s_\varepsilon) = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon},
\] (5.9)
where \(\phi_1(p) = C d_L(p, q_\varepsilon)^4\).

We next apply the definition of supersolutions and get
\[
\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon} + F(u(q_\varepsilon, t_\varepsilon), |\nabla_H \phi_2(q_\varepsilon)|) \geq 0,
\] (5.10)
where
\[
\phi_2(q) = -C d_L(p_\varepsilon, q)^4.
\]
By (5.9), in order to prove that \(\tilde{u}\) is a supersolution, we need to first substitute \(\nabla_H \phi_2(q_\varepsilon)\) in (5.10) with \(\nabla_H \phi_1(p_\varepsilon)\). By direct calculation, we have
\[
\nabla_H^p d_L(p, q)^4 = 4 \left(\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3, \delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3\right)
\]
and
\[
\nabla_H^q d_L(p, q)^4 = 4 \left(-\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3, -\delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3\right)
\]
with \(p = (x_p, y_p, z_p), q = (x_q, y_q, z_q)\) and
\[
\delta_1 = x_p - x_q, \ \delta_2 = y_p - y_q, \ \delta_3 = z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p,
\]
This reveals that
\[
|\nabla_H^p d_L(p, q)^4| = |\nabla_H^q d_L(p, q)^4|.
\]
In fact, we have
\[
|\nabla_H^p d_L(p, q)^4| = |\nabla_H^q d_L(p, q)^4| = 4d_L(p, q)^2(\delta_1^2 + \delta_2^2)^{\frac{1}{2}},
\]
which implies that \(|\nabla_H \phi_1(p_\varepsilon)| = |\nabla_H \phi_2(q_\varepsilon)|\) and their boundedness uniformly in \(\varepsilon\). Hence, due to (5.9), the equation (5.10) is now rewritten as
\[
\phi_i(p_\varepsilon, s_\varepsilon) + F_i(u(q_\varepsilon, t_\varepsilon), |\nabla_H \phi(p_\varepsilon, s_\varepsilon)|) \geq 0.
\]
Sending \(\varepsilon \to 0\), we obtain
\[
\phi_i(\hat{p}, \hat{t}) + F_i(\tilde{u}(\hat{q}, \hat{t}), |\nabla_H \phi(\hat{p}, \hat{t})|) \geq 0.
\]
In view of (5.7) and (5.8), we apply the assumption (A2) to get
\[
\phi_i(\hat{p}, \hat{t}) + F_i(\tilde{u}(\hat{q}, \hat{t}), |\nabla_H \phi(\hat{p}, \hat{t})|) \geq 0.
\]
and conclude the verification that $\tilde{u}$ is a supersolution. It follows that $\tilde{v} : \mathbb{H} \times [0, \infty) \to \mathbb{R}$, given by

$$\tilde{v}_k(p, t) = \tilde{u}_k(p, t) + 3L\delta^2 / 4,$$

for all $k = 1, 2, \ldots, m$, is also a supersolution of (1.1), as mentioned in Remark 2.5. Thanks to (5.1), we have $u_k(p, 0) \leq \tilde{v}_k(p, 0)$ for all $k$, which implies (5.5) by Theorem 3.1.

$$\square$$

It is however not clear whether the left-invariant Lipschitz continuity preserving property holds for the general parabolic system (1.1)–(1.2) without any symmetry assumption on the solution.

ACKNOWLEDGMENTS

This work was completed while the second author was visiting Fukuoka University, Japan. Its hospitality is gratefully acknowledged. The work of the first author was supported by JSPS Grant-in-Aid for Young Scientists, No. 16K17635. The work of the second author was supported by an AMS Simons Travel Grant.

REFERENCES


Qing Liu, Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan, qingliu@fukuoka-u.ac.jp

Xiaodan Zhou, Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA, xzhou3@wpi.edu